8. Quantity Calculations and Cost Estimates

The Bridge Section supports the project manager by providing several types of quantity calculations and cost estimates for structures during the planning, environmental, and design stages. These submissions are needed for planning and budgeting purposes.

8.1. Planning Stage (Level 1)

Bridge Section frequently provides planning estimates for projects involving bridges and other transportation structures.

Typically planning level estimates are requested at the Statewide Transportation Improvement Program (STIP) development and regional project initiation stage. These estimates are developed quickly, with very limited information, and require many assumptions. The following applies to developing a planning cost estimate:

8.1.8. Responsibility

Bridge Section engineers develop planning cost estimates. The Chief Bridge Engineer reviews all planning estimates before they are sent to the regions.

8.1.9. Basis for Estimate

Base the planning estimates on historical cost data, location, the anticipated structure and foundation type, and estimated square foot of deck area.

Deck area is a function of the anticipated bridge length and width. Develop an approximate length from any existing site information or by increasing the existing bridge (if any) length by 10 percent then rounding up to a logical value.

Consult with the Statewide Hydraulics Engineer to verify the existing waterway opening prior to finalizing the preliminary bridge length.

For bridge width, match the proposed roadway width plus the width of the proposed bridge rails/barriers. Widths are sometimes determined by reviewing other recent projects in the same highway vicinity of the project.

Use Table 8-1 as a starting point for estimating the bridge construction costs. For more complicated or difficult projects, the bridge engineer should use the higher end of the range.

8.1.10. Ancillary Costs

The values in Table 8-1 incorporate only the basic bid item costs for typical bridges. Increase these values to reflect the ancillary costs to determine the estimated total project cost.

Either the Bridge Section or the region will determine these costs. In either case, clearly identify in the estimate whether the ancillary costs are included. Use the following guidance for adding contingency, mobilization, and construction engineering costs to the planning cost estimate:

Bridge Approach Roadway Costs

Include the approach roadway costs (sometimes called “logical touchdown costs”) in planning estimates if not included separately by the regions. Use 30 percent of the cost of the bridge for the cost of transitioning from the new bridge to the existing roadway. In the Department’s experience, this provides a conservative estimate suitable for planning and preliminary budgeting purposes.

Contingencies

For anticipated but undetermined costs, add a 20 to 30 percent contingency factor based on the sum of the estimated construction costs. The contingency factor is related to the amount of uncertainty in the hydraulic, foundation, and roadway geometric information at the time the estimate is being developed.

Mobilization and Demobilization

Add 10 percent of the basic bid item costs for the contractor’s mobilization and demobilization. This is the cost incurred by the contractor to mobilize the labor and equipment necessary for construction. A higher percentage may be justified for projects in remote areas.

Engineering

Add 15 to 20 percent of the estimated cost for preliminary engineering. Add 15 percent for construction engineering.

Indirect Cost Allocation Program (ICAP)

Add the percentage specified in the most recent Department ICAP memorandum to the total preliminary engineering and construction costs to account for Department overhead cost to construct the project.

For smaller projects, percentages for contingencies, mobilization, and engineering may be higher. For larger projects, these percentages may be lower.

Table 8-1
2023 Structural Material Cost Ranges*

	Structure Types	Typical Span Range (feet)	Estimated Cost
Hydraulic Structures	Corrugated Pipe	1-7	$250-$500 / LF
	Concrete Box Culvert	5-20	$500 -$4000 / LF
	Plate Arch Culvert	10-20	$250-$750 / LF
Structures for Conventional Site Conditions	Post-Tensioned Concrete Box Girder	120-240	$300-$600 / SF
	Prestressed Concrete Voided Slab	20-60	$350-$750 / SF
	Prestressed Concrete Decked Bulb-Tee	50-145	$300-$500 / SF
	Prestressed Concrete Girder	50-140	$400-$600 / SF
	Steel Rolled Girder	20-120	$400-$600 / SF
	Steel Rolled Girder with Precast Deck Panels	20-120	$400-$750 / SF
	Steel Plate Girder	60-400	$400-$750 / SF
	Steel Box Girder	100-400	$400-$800 / SF
	Timber	10-20	$250-$500 / SF
	Glulam Timber	15-40	$250-$500 / SF
	Plate Arch Railroad Tube	25-35	$2000-$5000 / SF
Structures for Special Site Conditions	Segmental Post-Tensioned Box Girder	200-600	$750-$1500 / SF
	Cable Stayed Bridge	600-1200	$1000-$2000 / SF
	Suspension Bridge	600-5000	$1000-$3000 / SF
	Arch Bridge	50-400	$1000-$2000 / SF
	Movable Span Bridge	200-350	> $2500 / SF
	Tunnel	30+	> $3000 / SF

* *While these cost ranges are reasonably accurate at the time of publication, they should be used with caution due to inflation and volatility in construction materials, equipment operation, and labor costs.*

8.2. Bridge Type Selection (Level 2)

At the Bridge Type Selection stage, the project manager submits a "start package" including preliminary highway alignment data, proposed typical section, and site contours including right-of-way limits. Lay out the bridge using the preliminary roadway plan, profile, typical section, site topography, and other available design information included in the start package. The layout should include estimated piles sizes, cast-in-place (CIP) concrete member sizes, span lengths, and riprap geometry. Most member sizes can be accurately estimated from past projects, geometric compatibility, and simplified structural analysis.

The project manager may request estimates for multiple bridge types during the development of the environmental document.

Select the most feasible, constructable, and usually the most economical type of structure to fit the project and develop General Layout and Site Plan sheets. This structure may differ from the structure envisioned in earlier estimates. Use the General Layout and Site Plan to calculate preliminary quantities and develop the Bridge Type Selection estimate.

8.2.8. Quantity Calculations

General Guidelines

The quantity calculations require rapid (typically, a day per bridge or less) but close approximation of the final bridge quantities. Include all items required for bridge construction, and use graphs and tables, similar bridges, and computations based on dimensions from the preliminary plans. The objective is to complete a relatively accurate estimate in a minimal amount of time. Bridge quantities are typically placed in a table on the General Layout or Site Plan Sheet.

Consider the following guidelines when calculating quantities:

1. **Pay Items**. Each pay item has an official title and item number that is tied to the *Standard Specifications for Highway Construction*, which are also listed in the Department's bid item database. The Department uses these coded item numbers for tracking and as a historic database. Cross check all items against the *Standard Specifications*, Standard Modifications, and Special Provisions to ensure the use of the appropriate pay items, methods of measurement, and basis of payment. Division 500 presents structural items. For some specialty or new items, the pay item number may not be in the database. If unable to locate a pay item, the bridge engineer may request a new pay item through the Regional Bid Tab Coordinator.
2. **Units of Measurement**. Record the quantity of all contract bid items consistent with the terms and units of measurement presented in the applicable item sections of the *Alaska Standard Specifications for Highway Construction.* Table 8-2 presents the units of measurements for typical bridge items.
3. **Computations**. The bridge engineer may use manual and computer methods to compute bridge quantities. Prepare computation sheets for each bid item and retain all computation sheets in the project file.
4. **Significant Digits**. In all quantity calculations, retain enough significant digits so that accuracy is neither sacrificed nor exaggerated. Calculate quantities for individual structure elements to one significant figure beyond the rounding value shown in Table 8-2, when applicable.
5. **Rounding**. Match exactly the quantity of any item provided in quantity summaries with the number provided on computation sheets. Round the total item quantity for a structure to the value shown in Table 8-2. Note any required rounding of a raw estimate on the computation sheet consistent with Table 8-2. Do not round the result of a quantity calculation until the value is ready for incorporation into the Bridge Basis of Estimate Table. Do not round lump-sum items and items measured as "each."
6. **Bridge Basis of Estimate Table**. Segregate the quantities into separate columns with respect to the substructure and superstructure. For lump-sum items, provide the calculated quantity for the item.

Lump-Sum Items

In general, DOT&PF uses the lump-sum method of payment for items of work that are easily defined and unlikely to vary. Lump-sum items are estimated using the units identified in Table 8-2. When a lump-sum item is developed without using an estimating unit (Removal of Structures and Obstructions, Cofferdams, etc.), note the basis of the estimated cost and any special circumstances or relevant information in the quantity calculation package. Ensure this information is properly addressed in the plans and Special Provisions.

Excavation and Riprap

Where possible at the Bridge Type Selection stage, round excavation and riprap quantities to the nearest 10 cubic yards. Do not provide the dimensions of width, length, and depth more accurately than the nearest foot.

Concrete

Calculate concrete subtotals for various elements (e.g., abutments, piers) in cubic feet to the nearest tenth of a cubic foot. Carry dimensions to the hundredth of a foot, except for a thin cross section multiplied by a large length (e.g., slab cross section) where rounding to one hundredth could produce a large discrepancy in the final quantity. After totaling all subtotals, convert the total to cubic yards and round according to Table 8-2.

8.2.9. Cost Estimate

The bridge engineer updates the structural costs used for the planning construction cost estimate at the Bridge Type Selection stage. The bridge engineer should make a reasonable estimate of the structure quantities (within ± 10 percent). If the quantities can be estimated, use the following procedure for the updated cost estimate:

Unit Costs

Review historic data from similar projects to determine the unit costs. Possible sources include bid tabulations maintained by the Bridge Section and *Means Heavy Construction Cost Data*, an industry publication. For elements with little or no known information, contact industry sources for possible unit cost guidance.

See Table 8-2.

Adjustments

Adjust the estimated construction cost to reflect the conditions at the bridge site. Adjustment factors may include:

- Geographic location
- Age of most recent construction costs
- Recent trends in cost of materials (e.g., shortages)
- Extent of falsework required
- Anticipated difficulty of construction
- Size of project relative to size of previous projects for which cost data is available
- Known foundation problems at the bridge site
- Anticipated construction logistics (e.g., traffic control during construction)
- Any other factors that are considered appropriate for the structure
- Judgment and experience of the bridge engineer

Inflation

Do not adjust the cost for expected inflation at the time of construction. However, note the date the cost estimate was prepared.

Ancillary Costs

The discussion on ancillary costs found in Section 8.1 applies to the Bridge Type Selection cost estimate, except as follows:

- Reduce the percentage of anticipated but undetermined costs as more quantities are calculated.
- Add 15 percent of the estimated cost for construction engineering.

Preliminary engineering costs rarely vary enough to justify a separate line in a Bridge Type Selection estimate. In the unusual case where preliminary engineering costs are significantly different between alternatives, include separate line items for preliminary engineering and construction engineering.

Life-Cycle Costs Analysis

When required by the Chief Bridge Engineer, conduct a life-cycle cost analysis for each design alternative. This analysis allows the user to compare different design lives for each alternative. TRB's *National Cooperative Highway Research Program (NCHRP) Report 483: Bridge Life-Cycle Cost Analysis* contains a methodology for bridge life-cycle cost analysis (BLCCA), software that automates the methodology, and a guidance manual for implementing BLCCA.

Table 8-2
Quantity Calculations and Cost Estimates
(Type Selection)
Page 1 of 5

Pay Item Number	Pay Item Name	Quantity	Method of Calculating Quantities	Rounding Accuracy	Unit Cost
202.002 3.0000	Removal of Bridge	SF	Total area of existing bridge deck	1	$50/SF to $85/SF
205.000 1.0000	Excavation for Structures	CY	Only needed when excavation exceeds that associated with “normal” work. In such cases, assume that the total volume is equal to the cross sectional area at centerline roadway times the estimated length perpendicular to the cross section.	10	$45/CY to $75/CY
205.000 6.0000	Structural Fill	CY	Fill between the wingwalls times the maximum height from the bottom of the footing to the bottom of the approach slab or roadway section times 50 feet behind each backwall or end diaphragm.	1	$45/CY to $100/CY
501.000 1.0000	Class A Concrete	CY	Volume of concrete based upon preliminary dimensions of columns/piles, cap beams, wingwalls, diaphragms, etc. Assume all pipe piles are filled with CIP concrete from the bottom of the cap down 55 feet.	0.1	$1000/CY to $2500/CY
501.000 2.0000	Class A-A Concrete	CY	Volume of concrete based upon preliminary dimensions of CIP deck slabs and approach slabs.	0.1	$1500/CY to $2500/CY
501.000 7.0000	Precast Concrete Members, Girders	LBS	Determine the number and cross sectional area of prestressed girders (typically 6 to 8 girders in the cross section for standard two-lane bridges) using the Department’s bulb-tee program or other reasonable approach. Calculate the weight of each girder using 160 PCF.	1	$0.65/LB to $0.80/LB
503.000 1.0000	Reinforcing Steel	LBS	Assume 4% of the total weight of all CIP concrete (Class A + Class A-A) for standard decked bulb-tee girder bridge. Assume 8% when drilled shafts are used.	10	$2.00/LB to $2.75/LB

Table 8-2
Quantity Calculations and Cost Estimates
(Type Selection)

Pay Item Number	Pay Item Name	Quantity	Method of Calculating Quantities	Rounding Accuracy	Unit Cost
503.000 2.0000	Epoxy-Coated Reinforcing Steel	LBS	Assume 2% to 3% of the total weight of all CIP concrete (Class A + Class A-A) for standard decked bulb-tee girder bridge. Assume 6% when CIP decks are used.	10	\$2.75/LB to \$3.50/LB
504.000 1.0000	Structural Steel	LBS	Calculate the preliminary girder sized based upon standard depth-to-span ratio and slenderness ratios. In lieu of structural analysis, assume the following: Continuous spans: w = 20+0.2*L Simple spans: w = 25+0.3*L Where L is the maximum individual span length (feet) and w is the total unit weight of structural steel (pounds per square foot) including girders and all steel included in the pay item.	10	\$4.00/LB to \$6.50/LB \$10.00/LB for small quantities (e.g., seismic retrofit work)
505.000 5	Furnish Structural Steel Piles	LF	ABUTMENTS: For non-liquefiable soils, assume one HP 14 × 117 pile per girder reaction. For liquefiable soils, assume that one 24-inch diameter × ½ inch pipe pile is used per girder reaction. PIERS: For pier heights less than 15 feet, assume 36-inch diameter × ¾ inch pipe piles are used at a spacing of 10 feet. Otherwise, assume 48-inch diameter × 1-inch pipe piles are used at a spacing of 12 feet.	1	\$125/LF to \$175/LF for HP14X117 \$300/LF to \$400/LF for 36-inch diameter \$500/LF to \$750/LF for 48-inch diameter
505.000 6	Drive Structural Steel Piles	EA	The number of each type of pile to be driven	1	\$10,000/pile to \$15,000/pile for HP14x117 \$20,000/pile for 24-inch diameter \$30,000/pile to \$40,000/pile for 36-inch diameter \$40,000/pile to \$50,000/pile for 48-inch diameter

Table 8-2
Quantity Calculations and Cost Estimates
(Type Selection)
Page 3 of 5

Pay Item Number	Pay Item Name	Estimating Quantity	Method of Calculating Quantities	Rounding Accuracy	Unit Cost
507.000 1.0002	Steel Bridge Railing, 2-Tube	LF	Calculate the total two-tube bridge length including rail on the wingwalls from preliminary bridge layout	1	$250/LF to $300/LF
507.000 1.0003	Steel Bridge Railing, 3-Tube	LF	Calculate the total three-tube bridge length including rail on the wingwalls from preliminary bridge layout	1	$250/LF to $350/LF
507.000 4.0000	Concrete Barrier	LF	Calculate the total bridge barrier length including rail on the wingwalls from preliminary bridge layout	1	$250/LF to $325/LF
508.000 1.0000	Waterproofing Membrane	SF	Calculate total membrane area on bridge deck and approach slabs between curbs.	1	$10/SF to $15/SF
511.000 1.0000	Mechanically Stabilized Earth Wall	SF	Calculate the total area of wall face based upon preliminary bridge layout. Area of wall is measured from top of coping to bottom of wall panels.	1	$50/SF to $75/SF
512.000 2.0000	Falsework	SF	Typically the cost of falsework is included in the 501 pay items – that is, it is subsidiary to CIP concrete. Falsework may be bid as a separate pay item for CIP box girders or other non-standard bridge types are specified that require significant amounts of falsework. If falsework is bid as a separate pay item, reduce the unit cost of CIP concrete.	1	$50/SF to $100/SF
513.000 1.0000	Field Painting of Steel Structures	SF	Avg. value of 125 SF per ton of structural steel. Table 3 on page 270 of "Good Painting Practice Third Edition," Steel Structures Painting Council provides more detail. Calculate actual surface area and check conversion factors from this table on large, deep plate and box girders.	1	$35/SF to $50/SF for (E) bridges w/ zinc paint. $40/SF to $75/SF for (E) bridges w/ lead paint
513.000 2.0000	Field Metallizing of Steel Structures	SF	Calculate using the same methods as 513.0001.0000		$40/SF to $60/SF

Table 8-2
Quantity Calculations and Cost Estimates
(Type Selection)
Page 4 of 5

Pay Item Number	Pay Item Name	Estimating Quantity	Method of Calculating Quantities	Rounding Accuracy	Unit Cost
520.0001 .0000	Temporary Crossing	SF	Calculate the deck area based upon the estimated length and width required for the detour bridge. Typical bridge width is 34 feet out-to-out.	1	$125/SF to $175/SF
606.0016 .0000	Transition Rail	EA	Typically, there is one thrie beam transition railing at each end of the bridge railing – 4 per bridge.	1	$3000/EA to $4000/EA
611.0001 .0002	Riprap, Class II	CY	Calculate the volume of riprap using the preliminary bridge layout. The riprap volume can be estimated as the cross section area times the length of the riprap measured along the center-of-gravity of the cross section. In lieu of calculations, use 1000 CY for “normal” riprap usage and 2500 CY for “extensive” riprap usage.	10	$75/CY to $125/CY In remote locations, $250/CY depending upon local availability
631.0002 .0001	Geotextile, Erosion Control, Class 1	SY	Calculate the area of geotextile from the preliminary bridge layout. In lieu of calculations, use the same area of geotextile in SY as volume of riprap in CY.	10	$3.00/SY to $6.00/SY
640.0001 .0000	Mobilization and Demobilization	%	The mobilization and demobilization is typically taken to be 10% of the total construction cost. Thus, use 11.1% of the subtotal bridge bid cost.		Subtotal of all other bridge bid prices time 1/9
NA	Construction Engineering	%	Use 15% of the total estimated construction cost		Total bridge cost including mobilization times 15%
NA	ICAP	%	This “overhead” cost changes every year. Use the published value. In lieu of the published value, use 5%.		Total bridge plus construction engineering cost times ICAP %

Table 8-2
Quantity Calculations and Cost Estimates
(Type Selection)
Page 5 of 5

Pay Item Number	Pay Item Name	Estimating Quantity	Method of Calculating Quantities	Rounding Accuracy	Unit Cost
	Contingency	%	For preliminary designs, include a contingency that is reflective of the available information. Use the following: 1. 15% to 20% for 1R and 3R projects depending on the complexity and number of bridges 2. 20% to 25% when roadway, hydraulic and foundation information is available and used in the preliminary bridge layout 3. 25% to 30% when roadway information is available and an existing bridge is being replaced but no specific hydraulic or foundation data is available 4. 30% to 35% when roadway information is available but no hydraulic or foundation information is available and the proposed bridge is not reasonably close to any existing bridge		Total bridge cost including ICAP times the appropriate factor

8.3. PS&E Stage (Level 3)

8.3.8. Quantity Calculations

Update Calculations

The bridge engineer must revisit all quantity calculations as determined at the Bridge Type Selection stage (see Section 8.2.1) and update as necessary.

Reinforcing Steel

The reinforcing steel quantity is based on the weight of the reinforcing bar. Calculate bar lengths to hundredths of a foot. Accumulate total lengths of bar in each size before weights are extended to reduce rounding errors. Use Table 8-3 to obtain the weight in pounds for the appropriate reinforcing steel size. Do not include the weight of incidental items in this quantity.

Structural Steel

The structural steel quantity is based on the weight of the steel components in the structure. The contract documents present structural steel as a lump sum bid item, but also provide the steel weight as information only to assist the contractor in preparing the bid. Include the weight of all beams, plates, diaphragms, stiffeners, bearing plates, bolts and nuts, shear studs, rockers, rollers, pins and nuts, expansion dams, roadway drains and scuppers, weld metal and structural shapes for expansion joints, and pier protection in the estimate.

Calculate the weight of the steel using the following guidelines:

1. **Lengths.** Carry lengths to the hundredth of a foot.
2. **Unit Weights.** Structural steel has a weight density of 490 lb/feet^3. See the *Alaska Standard Specifications* for the weight density of other common metals.
3. **Shapes, Plates, Railing, and Flooring.** Calculate the nominal weights and dimensions as shown on the contract drawings, deducting for copes, cuts, and open holes, exclusive of bolt holes.
4. **Casting.** Compute the weights for castings from the dimensions shown on the plans with a 5 percent allowance for fillets and overruns. Deduct the weight for drillings or borings. The bridge engineer may use scale weights for castings of small complex parts, because it could be difficult to compute their weight accurately.
5. **Bolts, Nuts, and Washers.** Measure bolts, nuts, and washers for payment based on the computed weight as presented in the *Alaska Standard Specifications.*
6. **Fillet Welds.** Estimate fillet welds using Table 8-4 and the *Alaska Standard Specifications*.
7. **Rounding.** Round the total of the structural steel to the nearest 10 lbs.

8.3.9. Cost Estimate

Update Estimate

The bridge engineer must revise the cost estimate from the Bridge Type Selection stage (see Section 8.2.2) to reflect the final bid items and quantities. No "contingency" is used at the PS&E stage.

Submit the final estimate to the project manager in spreadsheet format along with the plans and Special Provisions. Some project managers also want a copy of the quantity calculations, and these should be submitted with the PS&E package if requested.

The project manager merges the cost estimates for structural and other project pay items to determine the Engineer's Estimate for the project.

Engineer's Estimate

The Engineer's Estimate is the final estimate used for programming construction funding. It lists the total quantity and estimated price for each pay item. Pay item quantities in the Engineer's Estimate must match the estimate of quantities in the plans and the bid schedule in the bidding documents. The project manager uses the Engineer's Estimate as a basis for requesting Authority to Advertise (ATA) a project. After bids are received, the Engineer's Estimate provides a basis for determining the reasonableness of the bids.

Design Aid Formulas

The following figures present various mathematical relationships to assist the bridge engineer with quantity estimating:

- Figure 8-1 — Slope Equations
- Figure 8-2 — Trigonometric Solution of Triangles
- Figure 8-3 — Area of Plane Figures
- Figure 8-4 — Surface Area and Volume of Solids

Table 8-3
Reinforcing Steel Sizes and Weights

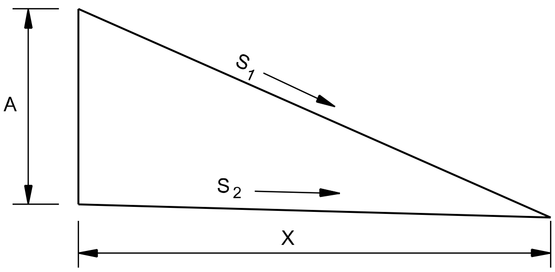
Bar Size		Nominal Properties		
		Diameter	Area	Weight
US Customary Designation	Metric Designation	(in)	(in^2)	(lb/ft)
#3	#10	0.375	0.11	0.376
#4	#13	0.500	0.20	0.668
#5	#16	0.625	0.31	1.043
#6	#19	0.750	0.44	1.502
#7	#22	0.875	0.60	2.044
#8	#25	1.000	0.79	2.670
#9	#29	1.128	1.00	3.400
#10	#32	1.270	1.27	4.303
#11	#36	1.410	1.56	5.313
#14	#43	1.693	2.25	7.650
#18	#57	2.257	4.00	13.600

Table 8-4
Weights of Fillet Welds

Nominal Size (in)	1/4	5/16	3/8	1/2	5/8	3/4	7/8	1
Weight (lb/ft)	0.20	0.25	0.35	0.55	0.80	1.10	1.50	2.00

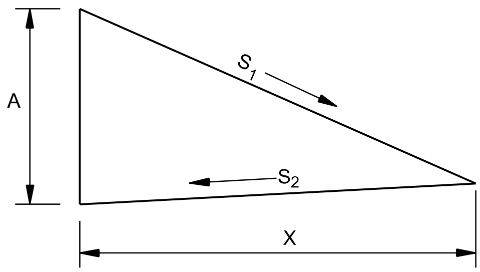
Note: Estimate other welds based on their theoretical dimension plus 50% for overrun.

CASE I



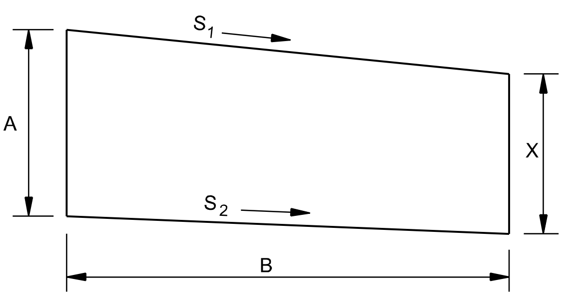
$$X = \frac{A}{S_1 - S_2}$$

CASE II



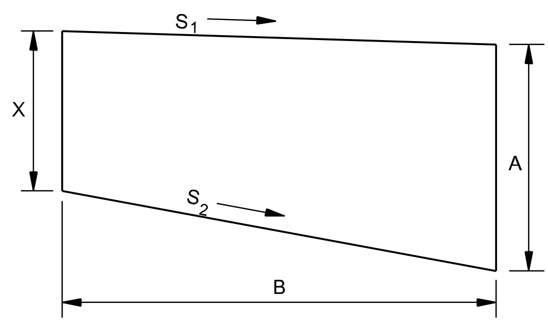
$$X = \frac{A}{S_1 + S_2}$$

CASE III



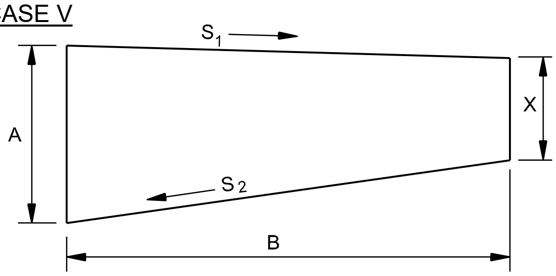
$X = A + (S_2 - S_1)\,B$ Area $= \frac{A + X}{2}\,(B)$

CASE IV



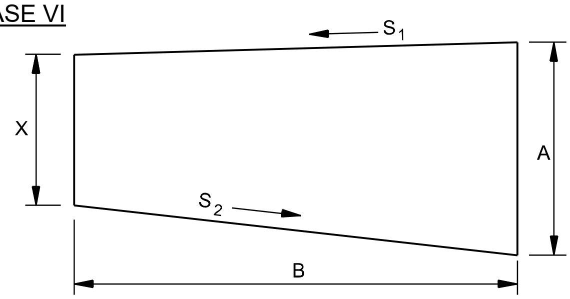
$X = A + (S_2 - S_1)\,B$ Area $= \frac{A + X}{2}\,(B)$

CASE V



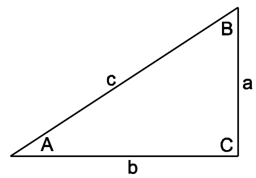
$X = A - (S_1 + S_2)\,B$ Area $= \frac{A + X}{2}\,(B)$

CASE VI



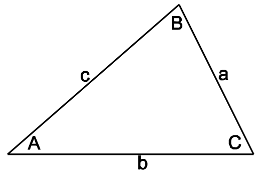
$X = A + (S_1 + S_2)\,B$ Area $= \frac{A + X}{2}\,(B)$

Figure 8-1
Slope Equations



Right-Angled Triangles		
Given:	**Sought:**	**Formulae:**
a,c	A,B,b	$\sin A = \frac{a}{c}$, $\cos B = \frac{a}{c}$, $b = \sqrt{c^2 - a^2}$
	Area	Area = $\frac{a}{2}\sqrt{c^2 - a^2}$
a,b	A,B,c	$\tan A = \frac{a}{b}$, $\tan B = \frac{b}{a}$, $c = \sqrt{a^2 + b^2}$
	Area	Area = $\frac{ab}{2}$
A,a	B,b,c	$B = 90° - A$, $b = a \cot A$, $c = \frac{a}{\sin A}$
	Area	Area = $\frac{a^2 \cot A}{2}$
A,b	B,a,c	$B = 90° - A$, $a = b \tan A$, $c = \frac{b}{\cos A}$
	Area	Area = $\frac{b^2 \tan A}{2}$
A,c	B,a,b	$B = 90° - A$, $a = c \sin A$, $b = c \cos A$
	Area	Area = $\frac{c^2 \sin A \cos A}{2}$ or $\frac{c^2 \sin 2A}{4}$

Figure 8-2
Trigonometric Solution of Triangles
Page 1 of 2



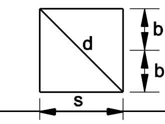
$$s = \frac{a+b+c}{2}$$

Oblique-Angled Triangles

Given:	Sought:	Formulae:
a,b,c	A	$\sin\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\cos\frac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}$, $\tan\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$
	B	$\sin\frac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{ac}}$, $\cos\frac{1}{2}B = \sqrt{\frac{s(s-b)}{ac}}$, $\tan\frac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$
	C	$\sin\frac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{ab}}$, $\cos\frac{1}{2}C = \sqrt{\frac{s(s-c)}{ab}}$, $\tan\frac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$
	Area	$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$
a,A,B	b,c	$b = \frac{a\sin B}{\sin A}$ $\quad c = \frac{a\sin C}{\sin A} = \frac{a\sin(A+B)}{\sin A}$
	Area	$\text{Area} = \frac{1}{2}ab\sin C = \frac{a^2\sin B\sin C}{2\sin A}$
a,b,A	B	$\sin B = \frac{b\sin A}{a}$
	c	$c = \frac{a\sin C}{\sin A} = \frac{b\sin C}{\sin B} = \sqrt{a^2+b^2-2(ab)(\cos C)}$
	Area	$\text{Area} = \frac{1}{2}(ab)(\sin C)$
$(2bc)(\cos A)$ a,b,C	A	$\tan A = \frac{a\sin C}{b-a\cos C}$ $\quad \tan\frac{1}{2}(A-B) = \frac{a-b}{a+b}\cot\frac{1}{2}C$
	c	$c = \sqrt{a^2+b^2-2ab\cos C} = \frac{a\sin C}{\sin A}$
	Area	$\text{Area} = \frac{1}{2}(ab)(\sin C)$
$a^2 = b^2 + c^2 - (2bc)(\cos A)$,	$b^2 = a^2 + c^2 - (2ac)(\cos B)$,	$c^2 = a^2 + b^2 - (2ab)(\cos C)$

Figure 8-2
Trigonometric Solution of Triangles
Page 2 of 2

Square



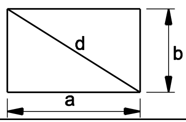
Diagonal = d = $s\sqrt{2}$

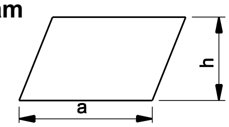
Area = $s^2 = 4b^2 = 0.5\ d^2$

Example: s = 6; b = 3; Area = $(6)^2 = 36$

d = 6 x 1.414 = 8.484

Rectangle and Parallelogram



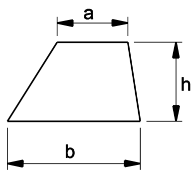


Area = ab or $b\sqrt{d^2 - b^2}$

Example: a = 6; b = 3

Area = 3 x 6 = 18

Trapezoid

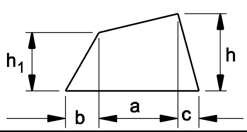


Area = $\frac{1}{2}h(a+b)$

Example: a = 2; b = 4; h = 3

Area = $\frac{1}{2} \times 3(2+4) = 9$

Trapezium



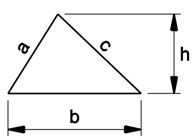
Area = $\frac{1}{2}\left[a(h+h_1) + bh_1 + ch\right]$

Example: a = 4; b = 2; c = 2; h = 3; h_1 = 2

Area = $\frac{1}{2}\left[4(3+2)+(2\times 2)+(2\times 3)\right] = 15$

Triangles

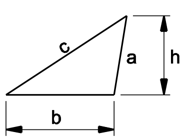
Formulas apply to both figures.



Area = $\frac{1}{2}$ bh

Example: h = 3; b = 5

Area = $\frac{1}{2}(3\times 5) = 7.5$

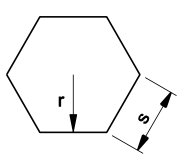


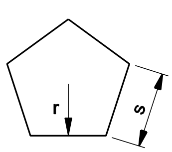
Area = $\sqrt{s(s-a)(s-b)(s-c)}$ where: s = $\frac{a+b+c}{2}$

Example: a = 2; b = 3; c = 4

s = $\frac{2+3+4}{2}$ = 4.5; Area = $\sqrt{4.5(4.5-2)(4.5-3)(4.5-4)} = 2.9$

Regular Polygons





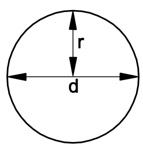
$$\text{Area}\begin{cases} 5\text{ sides} = 1.720477\ s^2 = 3.63271\ r^2 \\ 6\text{ sides} = 2.598150\ s^2 = 3.46410\ r^2 \\ 7\text{ sides} = 3.633875\ s^2 = 3.37101\ r^2 \\ 8\text{ sides} = 4.828427\ s^2 = 3.31368\ r^2 \\ 9\text{ sides} = 6.181875\ s^2 = 3.27573\ r^2 \\ 10\text{ sides} = 7.694250\ s^2 = 3.24920\ r^2 \\ 11\text{ sides} = 9.365675\ s^2 = 3.22993\ r^2 \\ 12\text{ sides} = 11.196300\ s^2 = 3.21539\ r^2 \end{cases}$$

n = number of sides; r = short radius; s = length of side; R = long radius.

Area = $\frac{n}{4}s^2\cot\frac{180°}{n} = \frac{n}{2}R^2\sin\frac{360°}{n} = nr^2\tan\frac{180°}{n}$

Figure 8-3
Area of Plane Figures
Page 1 of 3

Circle



$\pi = 3.1416$; A = area; d = diameter

p = circumference or periphery; r = radius

$$p = \pi d = 3.1416d \qquad p = 2\sqrt{\pi A} = 3.54\sqrt{A}$$

$$p = 2\pi r = 6.2832r \qquad p = \frac{2A}{r} = \frac{4A}{d}$$

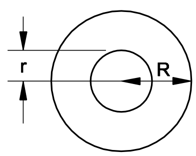
$$d = \frac{p}{\pi} = \frac{p}{3.1416} \qquad d = 2\sqrt{\frac{A}{\pi}} = 1.128\sqrt{A}$$

$$r = \frac{p}{2\pi} = \frac{p}{6.2832} \qquad r = \sqrt{\frac{A}{\pi}} = 0.564\sqrt{A}$$

$$A = \frac{\pi d^2}{4} = 0.7854d^2 \qquad A = \frac{p^2}{4\pi} = \frac{p^2}{12.57}$$

$$A = \pi r^2 = 3.1416r^2 \qquad A = \frac{pr}{2} = \frac{pd}{4}$$

Circular Ring



Area = $\pi\left(R^2 - r^2\right) = 3.1416\left(R^2 - r^2\right)$

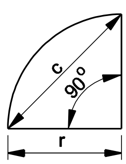
Area = $0.7854\ (D^2 - d^2) = 0.7854(D - d)(D + d)$

Area = difference in areas between the inner and outer circles.

Example: R = 4; r = 2

Area = $3.1416(4^2 - 2^2) = 37.6992$

Quadrant

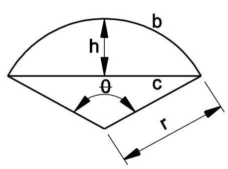


Area = $\dfrac{\pi r^2}{4} = 0.7854\,r^2 = 0.3927c^2$

Example: r = 3; c = chord

Area = $0.7851 \times 3^2 = 7.0686$

Segment



b = length of arc; θ = angle in degrees; c = chord = $\sqrt{4\left(2hr - h^2\right)}$

Area = $\dfrac{1}{2}\left[br - c(r - h)\right] = \pi r^2\,\dfrac{\theta}{360} - \dfrac{c(r - h)}{2}\,p$

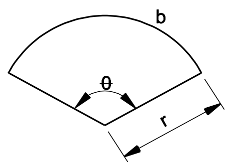
When θ is greater than 180°, then $\dfrac{c}{2}$ x difference between r and h is added to the fraction $\dfrac{\pi r^2 \theta}{360}$

Example: r = 3; θ = 120°; h = 1.5

Area = $3.1416 \times 3^2 \times \dfrac{120}{360} - \dfrac{5.196\,(3 - 1.5)}{2} = 5.5278$

Figure 8-3
Area of Plane Figures
Page 2 of 3

Sector



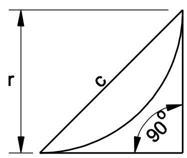
$$\text{Area} = \frac{br}{2} = \pi r^2 \frac{\theta}{360°}$$

θ = angle in degrees; b = length of arc

Example: r = 3; θ = 120°

$$\text{Area} = 3.1416 \times 3^2 \times \frac{120}{360} = 9.4248$$

Spandrel

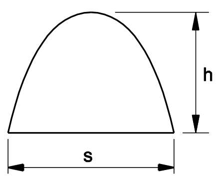


Area = 0.2146 r^2 = 0.1073 c^2

Example: r = 3

Area = 0.2146 x 3^2 = 1.9314

Parabola



l = length of curved line = periphery – s

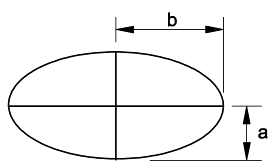
$$l = \frac{s^2}{8h}\left[\sqrt{c(1+c)} + 2.0326 \text{ x } \log\left(\sqrt{c} + \sqrt{1+c}\right)\right] \text{ where } c = \left(\frac{4h}{s}\right)^2$$

$$\text{Area} = \frac{2}{3} sh$$

Example: s = 3; h = 4

$$\text{Area} = \frac{2}{3} \times 3 \times 4 = 8$$

Ellipse



Area = $\pi ab = 3.1416ab$

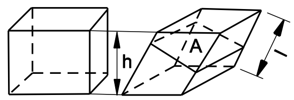
$$\text{Circumference} = 2\pi\sqrt{\frac{a^2 + b^2}{2}} \text{ (close approximation)}$$

Example: a = 3; b = 4.

Area = 3.1416 x 3 x 4 = 37.6992

$$\text{Circumference} = 2 \text{ x } 3.1416\sqrt{\frac{(3)^2 + (4)^2}{2}} = 6.2832 \text{ x } 3.5355 = 22.21$$

Figure 8-3
Area of Plane Figures
Page 3 of 3

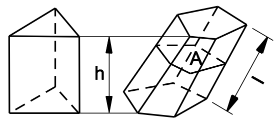


Parallelopiped

S = perimeter (P) perpendicular to sides x lateral length (l): Pl

V = area of base (B) x perpendicular height (h): Bh

V = area of section (A) perpendicular to sides x lateral length (l): Al

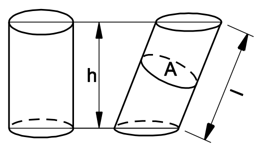


Prism, Right, or Oblique, Regular or Irregular

S = perimeter (P) perpendicular to sides x lateral length (l): Pl

V = area of base (B) x perpendicular height (h): Bh

V = area of section (A) perpendicular to sides x lateral length (l): Al



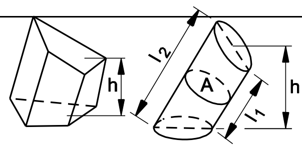
Cylinder, Right, or Oblique, Circular or Elliptic, etc.

S = perimeter of base (P) x perpendicular height (h): Ph

S = perimeter (P_1) perpendicular to sides x lateral length (l): P_1l

V = area of base (B) x perpendicular height (h): Bh

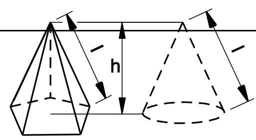
V = area of section (A) perpendicular to sides x lateral length (l): Al



Frustum of any Prism or Cylinder

V = area of base (B) x perpendicular distance (h) from base to center of gravity of opposite face: Bh

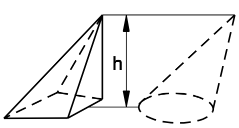
For cylinder: $\frac{1}{2} A \left(l_1 + l_2\right)$



Pyramid or Cone, Right and Regular

S = perimeter of base (P) x $\frac{1}{2}$ slant height (l): $\frac{1}{2}$Pl

V = area of base (B) x $\frac{1}{8}$ perpendicular height (h): $\frac{1}{8}$Bh

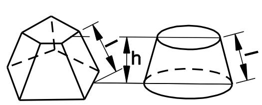


Pyramid or Cone, Right, or Oblique, Regular or Irregular

V = area of base (B) x $\frac{1}{3}$ perpendicular height (h): $\frac{1}{3}$Bh

V = $\frac{1}{3}$ volume of prism or cylinder of same base and perpendicular height

V = $\frac{1}{2}$ volume of hemisphere of same base and perpendicular height



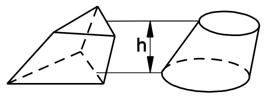
Frustum of Pyramid or Cone, Right and Regular, Parallel Ends

S = (sum of perimeter of base (P) and top (p)) x $\frac{1}{2}$ slant height (l): $\frac{1}{2}$l (P + p)

V = (sum of areas of base (B) and top (b) + square root of their products) x $\frac{1}{3}$ perpendicular height (h): $\frac{1}{3} h \left(B + b + \sqrt{Bb}\right)$

S = Lateral or Convex Surface Area *V = Volume*

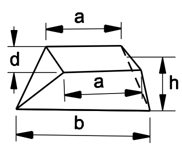
Figure 8-4
Surface Area and Volume of Solids
Page 1 of 3



Frustum of any Pyramid or Cone, Parallel Ends

V = (sum of areas of base (B) and top (b) + square root of their products)

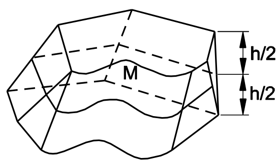
x $\frac{1}{8}$ perpendicular height (h): $\frac{1}{8}\,h\left(B + b + \sqrt{Bb}\right)$



Wedge, Parallelogram Face

V = $\frac{1}{6}$ (sum of three edges (a, b, a) x perpendicular height (h)

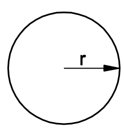
x perpendicular width (d): $\frac{1}{6}\,d\,h\,(2a+b)$



Prismatoid

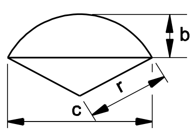
V = $\frac{1}{6}$ perpendicular height (h) x (sum of areas of base (B) and top (b) + 4 x area of section (M) parallel to bases and midway between them): $\frac{1}{6}\,h(B + b + 4M)$

The Prismatoid formula applies also to any of the foregoing solids with parallel bases, to pyramids, cones and spherical sections, and to many solids with irregular surfaces.



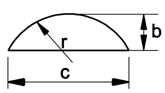
Sphere

$S = 4\,\pi\, r^2 = \pi\, d^2 = 3.14159265\, d^2$ $\qquad$ $V = \frac{4}{3}\pi\, r^3 = \frac{1}{6}\pi\, d^3 = 0.52359878\, d^3$



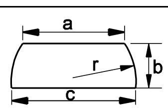
Spherical Sector

$S = \frac{1}{2}\pi\, r(4b + c)$ $\qquad$ $V = \frac{2}{3}\pi\, r^3 b$



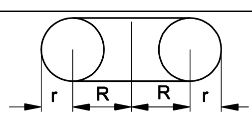
Spherical Segment

$S = 2\,\pi\, r\, b = \frac{1}{4}\pi\left(4b^2 + c^2\right)$ $\qquad$ $V = \frac{1}{3}\pi\, b^2\,(3r - b) = \frac{1}{24}\pi\, b\left(3c^2 + 4b^2\right)$



Spherical Zone

$S = 2\,\pi\, r\, b$ $\qquad$ $V = \frac{1}{24}\pi\, b\left(3a^3 + 3c^2 + 4b^2\right)$



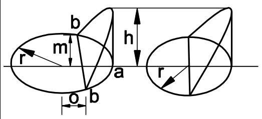
Circular Ring

$S = 4\,\pi^2 Rr$ $\qquad$ $V = 2\,\pi^3\, Rr^2$

S = Lateral or Convex Surface Area $\qquad$ *V = Volume*

Figure 8-4
Surface Area and Volume of Solids
Page 2 of 3

Ungula of Right, Regular Cylinder



Base = Segment, b a b

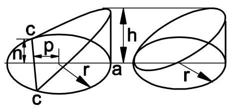
$S = (2\,r\,m - o \times \text{arc, } b\,a\,b)\ \dfrac{h}{r - o}$

$V = \left(\dfrac{2}{3} m^3 - o \times \text{area, } b\,a\,b\right) \dfrac{h}{r - o}$

Base = Half Circle

$S = 2rh$

$V = \dfrac{2}{3} r^2\, h$



Base = Segment, c a c

$S = (2rn + p \times \text{arc, } c\,a\,c)\ \dfrac{h}{r + p}$

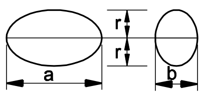
$V = \left(\dfrac{2}{3} n^3 + p \times \text{area, } cac\right) \dfrac{h}{r + p}$

Base = Circle

$S = r\,\pi\, h$

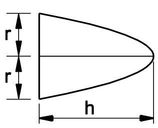
$V = \dfrac{1}{2} r^2\, \pi\, h$

Ellipsoid



$V = \dfrac{1}{3} \pi\, r\, a\, b$

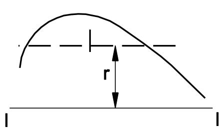
Paraboloid



$V = \dfrac{1}{2} \pi\, r^2 h$

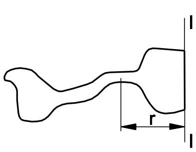
Ratio of corresponding volumes of a Cone, Paraboloid, Sphere, and Cylinder of equal height: $\dfrac{1}{3} : \dfrac{1}{2} : \dfrac{2}{3} : 1$

Bodies Generated by Partial or Complete Revolution



$\left.\begin{array}{l} l = \text{length of a curve} \\ A = \text{area of a plane} \end{array}\right\}$ rotating about an axis l - l on one side and in plane of axis

r = distance of center of gravity of line or plane from axis l - l and for any angle of revolution, a°



$\dfrac{2\,r\,\pi\,a°}{360}$ = length of arc described by center of gravity

S = length of curve x length of arc about axis = $l\ \dfrac{2\,r\,\pi\,a°}{360}$

For complete revolution, $S = 2\,r\,\pi\, l$

V = area of plane x length of arc about axis = $A\ \dfrac{2\,r\,\pi\,a°}{360}$

For complete revolution, $V = 2\,r\,\pi\, A$

S = Lateral or Convex Surface Area *V = Volume*

Figure 8-4
Surface Area and Volume of Solids
Page 3 of 3